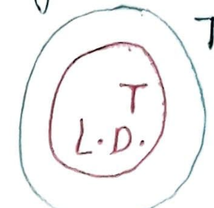
Th:- 4. Any superset of L.D. set of vectors is also L.D.



Proof:- let $T = \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is the L.D. set of vectors of vector space V(F).

$\exists$ scalars $a_1, a_2, \dots\dots, a_n \in F$ s.t.

$a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0 \quad \text{—(i)} \Rightarrow$ at least one of the scalar is non-zero.

{or $a_1, a_2, \dots a_n$ are all not zero}

Let $T_1 = \{\alpha_1, \alpha_2, \dots, \alpha_n, \alpha_{n+1}, \alpha_{n+2}, \dots, \alpha_{n+m}\}$

is any superset of T.

We have to prove that T_1 is also L.D.

By eq.^{n} (i), $a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n = 0$

$\Rightarrow \underbrace{a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n}_{} + 0.\alpha_{n+1} + 0.\alpha_{n+2} + \dots + 0.\alpha_{n+m}$

$= 0 + 0 + 0 + \dots + 0$

$= 0$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_n\alpha_n + 0.\alpha_{n+1} + \dots + 0.\alpha_{n+m} = 0$

$\Rightarrow a_1, a_2, \dots a_n$ are all not zero (at least one of the scalar $a_1, a_2, \dots a_n$ is non zero) (by (i))

$\Rightarrow T_1$ is L.D. set.

$\Rightarrow$ Every superset of L.D. set of vectors is also L.D.

H.P.

Th:- If $V(F)$ be a vector space and T is a subset of some non zero vectors of vector space, Then T is L.D. iff ~~some~~ some vector α_j of T can be Expressed as a linear combination of preceding one's, where $2 \le j \le n$. (6)

Proof:- (Necessary Condition) ($\Rightarrow$)

let $T = \{\alpha_1, \alpha_2, \ldots, \alpha_n\}$ is a L.D. set of vectors. $\because \alpha_1 \neq 0$ (given)

$\Rightarrow a_1 \alpha_1 = 0 \Rightarrow a_1 = 0$

$\Rightarrow \{\alpha_1\}$ is L.I. set

for the set $\{\alpha_1, \alpha_2\}$, if this set is L.D. then $\exists$ scalars $a_1, a_2 \in F$ s.t. $a_1\alpha_1 + a_2\alpha_2 = 0$ —(i)

where a_1, a_2 are non-zero scalars

$\{ \because$ if $a_1 = 0$ then from Eq^n (i) $a_2\alpha_2 = 0 \Rightarrow a_2 = 0$ $(\alpha_2 \neq 0)$

similarly if $a_2 = 0 \Rightarrow a_1 = 0$ (as $\alpha_1 \neq 0$) $\}$

$\Rightarrow a_1\alpha_1 + a_2\alpha_2 = 0 \Rightarrow \alpha_2 = -a_1 a_2^{-1} \alpha_1$

$\{ a_1 \in F, a_2 \in F \Rightarrow a_1 \in F, a_2^{-1} \in F \Rightarrow a_1 a_2^{-1} \in F$ by defin. of F $\}$

$\Rightarrow \alpha_2$ can be expressed as a L.C. of its preceding vector α_1. Now if the set $\{\alpha_1, \alpha_2\}$ is L.I then ~~Similarly~~ taking the set $\{\alpha_1, \alpha_2, \alpha_3\}$ of vectors. ~~is L.D.~~

If this set is L.D.

then the vector α_3 can be expressed as a (7)
L.C. of its preceding vectors
$\{ a_1\alpha_1 + a_2\alpha_2 + a_3\alpha_3 = 0 \Rightarrow \alpha_3 = -a_1 a_3^{-1}\alpha_1 - a_2 a_3^{-1}\alpha_2 \}$

But if the set of vectors $\{\alpha_1, \alpha_2, \alpha_3\}$ is L.I.
then we take next set of vectors $\{\alpha_1, \alpha_2, \alpha_3, \alpha_4\}$
and repeat the entire process.
Finally let for $j \in N$ where $2 \leq j \leq n$ & the
set $\{\alpha_1, \alpha_2, \cdots, \alpha_{j-1}\}$ of vectors is L.I.; then
Taking the set of vectors $\{\alpha_1, \alpha_2, \cdots, \alpha_{j-1}, \alpha_j\}$.
If this set is L.D. then

$$a_1\alpha_1 + a_2\alpha_2 + \cdots + a_j\alpha_j = 0 \quad \text{——} (2)$$

the scalar $a_1, a_2, \cdots a_j \in F$
where at least one of
is non zero.

Here $a_j \neq 0$ ~~$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots +$~~

as if $a_j = 0 \Rightarrow a_1\alpha_1 + a_2\alpha_2 + \cdots + a_{j-1}\alpha_{j-1} + 0.\alpha_j = 0$ (by (2))
where at least one $a_1, a_2, \cdots a_{j-1}$ is non-zero
$\Rightarrow \{\alpha_1, \alpha_2, \cdots \alpha_{j-1}\}$ is L.D. set
which is contradiction of our
basic assumption

$\Rightarrow a_j \neq 0$ by Eq.^{n} (2)

$$a_j\alpha_j = -a_1\alpha_1 - a_2\alpha_2 \cdots - a_{j-1}\alpha_{j-1}$$

$$\Rightarrow \alpha_j = -a_1 a_j^{-1}\alpha_1 - a_2 a_j^{-1}\alpha_2 - \cdots - a_{j-1} a_j^{-1}\alpha_{j-1}$$

$(\because a_i \in F,\ a_j^{-1} \in F \Rightarrow a_i a_j^{-1} \in F \Rightarrow -a_i a_j^{-1} \in F\}$ (8)

$$\Rightarrow \alpha_j = \sum_{i=1}^{j-1} (-a_i a_j^{-1})\, \alpha_i$$

$\Rightarrow$ α_j can be expressed as a L.C. of its preceding vectors.

sufficient condition ($\Leftarrow$) Let the vectors α_j can be expressed as a L.C. of its preceding vectors.

$$\Rightarrow \alpha_j = a_1\alpha_1 + a_2\alpha_2 + \dots + a_{j-1}\alpha_{j-1}$$

where $a_1, a_2, \dots a_{j-1} \in F$

$$\Rightarrow a_1\alpha_1 + a_2\alpha_2 + \dots + a_{j-1}\alpha_{j-1} + (-1)\alpha_j = 0$$

$\Rightarrow$ set $\{\alpha_1, \alpha_2, \dots, \alpha_{j-1}, \alpha_j\}$ of vectors is L.D. as the scalar coefficient of α_j is non-zero.

$\Rightarrow \{\alpha_1, \alpha_2, \dots \alpha_n\}$ is also L.D. set of vectors as every superset of L.D. set of vectors is also L.D. (Th. 4)

H.P.